Phase effect of two coupled periodically driven Duffing oscillators

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We consider the effect of phase difference in mutually coupled chaotic oscillators with a large coupling strength. The phase difference destroys the synchronization of chaotic oscillators, and lag synchronization is observed. For large difference, it even takes the coupled oscillators from chaotic motion to regular motion. For small frequency detuning of two driving forces, stochastic breathing, bifurcation delay, and stochastic transition are observed. The mutually coupled periodically driven Duffing oscillators are taken as a numerical example. [S1063-651X(98)05911-X]

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I. INTRODUCTION

The phenomenon of synchronization of coupled chaotic system has recently become a great topic of interest. Many situations can be represented in terms of coupled nonlinear such as optics [1], Josephson-junction arrays [2], condensed-matter physics [3], chemical reaction [4], and biology [5]. The dynamics of two coupled maps has been reported [6]. For a diffusely coupled autonomous continuous-time system, The chaos synchronization composed of identical chaotic oscillators has been studied, the general conditions for stability of the synchronized state was derived [7]. The two symmetrically coupled resonators are found to display period doubling, Hopf bifurcations, entrainment horns, and breakup of the torus [8]. A system of two coupled van der Pol oscillators shows a rich fractal structure when several attractors coexist [9]. Bifurcation diagrams and phase diagrams of two coupled periodically driven identical Duffing oscillators were presented [10]. Intermingled basins in nonlinearly coupled Duffing oscillators have been reported [11]. In these works, the frequency and the phase difference of external forces are identical. The synchronization of chaotic oscillators, both theoretically and in analog electronic circuits, has been investigated [12]. The new effect of phase synchronization of weakly coupled self-sustained chaotic oscillators has been found [13], because in these systems the phase is free, and therefore can be adjusted by small coupling. Therefore, chaos synchronization in mutually coupled autonomous systems includes two transitions. First, phase synchronization (no threshold for coupling strength) appears. Second, further increase of coupling induces the chaotic amplitudes' coincidence.

Many previous works focus on autonomous differential equations. For nonautonomous chaotic oscillators, Carroll and Pecora have shown that correcting the phase of the forcing term can allow the response system to synchronize with the driven system [14]. However, for driven systems, in the typical case, the phase and frequency of these two systems are often not in coincidence. Therefore, the phase difference between the two forces may effect chaos synchronization. Recently, it has been shown that the phase difference of two externally driven forces can play an important role in the driven systems. For example, in the case of suppression of chaos by means of weak parametric modulations in the forced pendulum, a suitable initial phase difference between the two modulations can suppress chaos using very small amplitude [15,16]. To study the effects of the phase and the small detuning of two frequencies in the externally driven oscillators, we consider two mutually coupled Duffing oscillators. The driven double-well Duffing oscillator has been carefully investigated [17]. Bifurcation diagrams and phase diagrams of the two-coupled periodically driven *identical* Duffing oscillators have been investigated [18].

The single driven Duffing oscillator is expressed by

$$\ddot{x}+\alpha\dot{x}-x+x^3=\beta\cos(\omega t), \tag{1}$$

where α is the damping parameter, and the other two parameters β and ω are the amplitude and the frequency of an external driving force, respectively. For $\alpha=0.1$, $\beta=0.3$, and $\omega=1$, this equation has two attractors, with one being chaotic and the other being large stable 1-periodic for different initial conditions [19]. Here, we just consider the chaotic attractor. For very small β, Eq. (1) has two period-1 orbits, one lying in the region $x>0$, which we denote P^+, and the other in the region $x<0$, which we denote P^- (see Fig. 9). Written as a set of differential equation, Eq. (1) has the form

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=-\alpha y+x-x^3+\beta\cos(\omega t).\end{aligned} \tag{2}$$

The mutually coupled Duffing oscillator considered in the paper is written as

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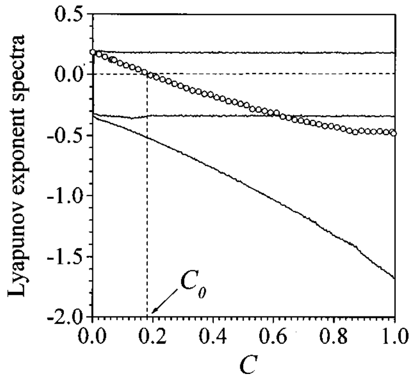

FIG. 1. The Lyapunov exponent (LE) spectra (lines) and the largest transversal LE (open circles) vs coupling strength C for $\varphi=0$. It indicates that, at $C>C_0\approx0.19$, chaos synchronization occurs, and the coupled system turns from hyperchaos (with two positive LEs) to chaos. The Largest Lyapunov exponent always remains at the same value.

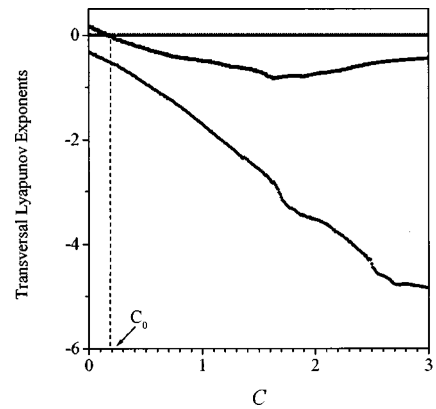

FIG. 2. The transversal Lyapunov exponent (TLE) spectra vs coupling strength C for $\varphi=0.2\pi$. It indicates that, at $C>C_0\approx0.19$, The largest TLE turn to zero.

$$\dot{x}=y+C(u-x), \tag{3a}$$

$$\dot{y}=-\alpha y+x-x^3+\beta\cos(\omega t), \tag{3b}$$

$$\dot{u}=\nu+C(x-u), \tag{3c}$$

$$\dot{\nu}=-\alpha\nu+u-u^3+\beta\cos[(\omega+\Delta\omega)t+\varphi], \tag{3d}$$

where φ and C are the initial phase difference of the two oscillators and the coupling strength, respectively, $0\leqslant\varphi\leqslant2\pi$. $\Delta\omega$ is a frequency detuning. Our discussion is based on a Poincare map of Eq. (3) strobed at times $t_n=n2\pi/\omega$, here n is a positive integer.

The paper is organized as follows: Section II discusses the effect of phase φ for $\Delta\omega=0$. Section III gives the results of the small frequencies detuning. Finally, Sec. IV includes a summary of the results and conclusions.

II. EFFECT OF THE PHASE DIFFERENCE

In this case, $\Delta\omega=0$, so two coupled Duffing oscillators have same frequency ω. In what follows, we fix $\omega=1$. When coupling strength C is zero, the amplitudes of two chaotic oscillators are different because of the sensitive dependence on the initial conditions. When $\varphi=0$ and $C\geqslant C_0$ (≈0.19) (Fig. 1), the largest transversal Lyapunov exponent is negative, and the two chaotic oscillators can be completely synchronized, which means $x(t)=u(t)$. For the case that the initial phase difference φ is not zero ($\varphi=0.2\pi$, see Fig. 2), the C_0 also is about 0.19, which is the same as $\varphi=0$. Compare Fig. 1 and Fig. 2; it indicates that the coupling strength just overcomes the initial sensitivity of chaotic orbits for the forced driven systems. Because the phase of these systems is not free, the two output signs contain a phase difference. On the contrary, for autonomous continuous systems, when the coupling strength is above a certain critical value, the coupled systems arrive at complete synchronization. Phase synchronization has occurred in this progress because it has no threshold [13]. However, if the initial phase difference φ is not zero (Fig. 2), even for large C, the complete synchronization of two oscillators is destroyed [Fig. 3(b)]. To understand this result, we now discuss the synchronization manifold $x=u$. Let $d_x=x-u$, $d_y=y-\nu$, and suppose $x\approx u$, then we can get

$$\begin{aligned}\dot{d}_x&=d_y-2Cd_x,\\ \dot{d}_y&=-\alpha d_y+d_x-3x^2-\beta\sin(\varphi/2)\sin(\omega t+\varphi/2).\end{aligned} \tag{4}$$

For $\varphi=0$, the sin term vanishes; then when $C\geqslant C_0$, d_x and d_y tend to zero and a complete synchronous state occurs. If φ is not zero, there always exists a zero Lyapunov exponent in Eq. (4); d_x and d_y do not tend to zero.

Figure 3(a) shows a bifurcation diagram for coupling strength $C=1$ versus the initial phase φ. To quantitatively determine the level of the mismatch of chaos synchronization, we use a similarity function $S(\tau)$ as a time averaged difference between the variables x and u taken with the time shift τ [20],

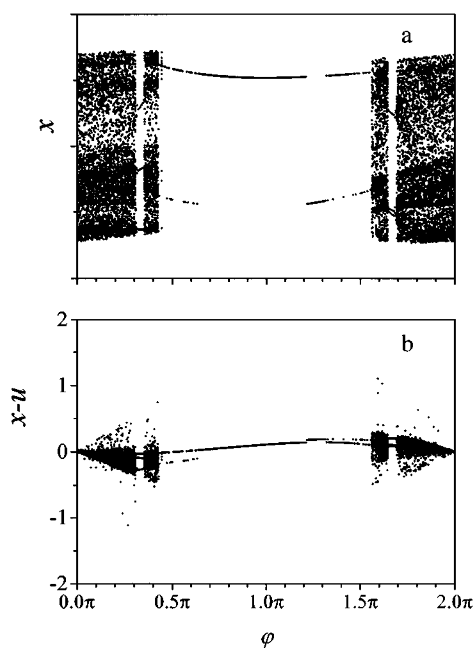

FIG. 3. Bifurcation diagram for large coupling strength $C=1$ showing the Poincare map vs the initial phase difference φ. At $\varphi\approx0.3\pi$ and $\varphi\approx0.4\pi$ periodic orbits occur. (a) x vs φ. (b) $x-u$ vs φ. This indicates that the synchronous state occurs at $\varphi=0$.

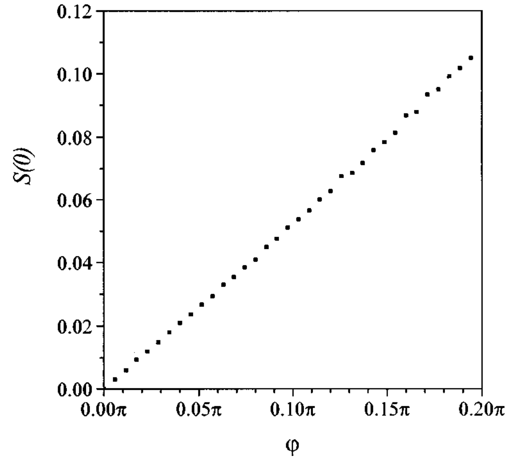

FIG. 4. The similarity function $S(0)$ vs the initial phase difference φ for $C=1$. For complete synchronization, $S(0)=0$. The increase of $S(0)$ with φ is linear.

$$S^2(\tau)=\frac{\langle [x(t+\tau)-u(t)]^2\rangle}{[\langle x^2(t)\rangle\langle u^2(t)\rangle]^{1/2}},\tag{5}$$

and plot the similarity function $S(0)$ versus φ; the result is plotted in Fig. 4 for $C=1$. The increase of the mismatch with the phase difference is linear for small φ. Figure 5 shows the similarity functions $S(\tau)$ for various coupling strengths. For strong coupling strength ($C>C_0$), we can observe that $S_{\min}$ [a minimum of $S(\tau)$] appears. It indicates the existence of some characteristic time shift τ_0 between $x(t)$ and $u(t)$. As C tends to infinity, $S_{\min}$ and τ_0 tend to zero. In this case, two oscillators are complete synchronization.

Figure 6 is the Lyapunov exponent spectra corresponding to Fig. 3. It indicates that, at $\varphi\approx 0.30\pi$, and $\varphi\approx 0.44\pi$, the chaotic oscillation turn out to be periodic, or *vice versa*. For stronger coupling strength $C=5$ we obtain Fig. 7. Compare it with Fig. 3; we can find that the two results have very similar bifurcation diagrams, and $x(t)-u(t)$ is effectively suppressed. To quantitatively describe the relationship between the coupling strength and the level of chaos desynchronization, we calculate the similarity function $S(0)$ versus C, and set $\varphi=0.2\pi$. The result shows in Fig. 8. With

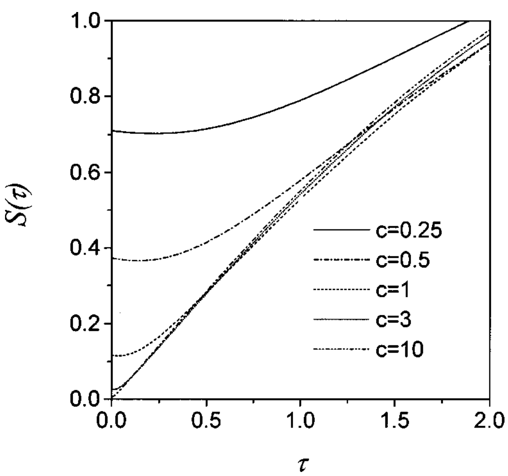

FIG. 5. Similarity function $S(\tau)$ for different values of coupling strength C. When $C>C_0$, $S_{\min}$, a minimum of $S(\tau)$, appears.

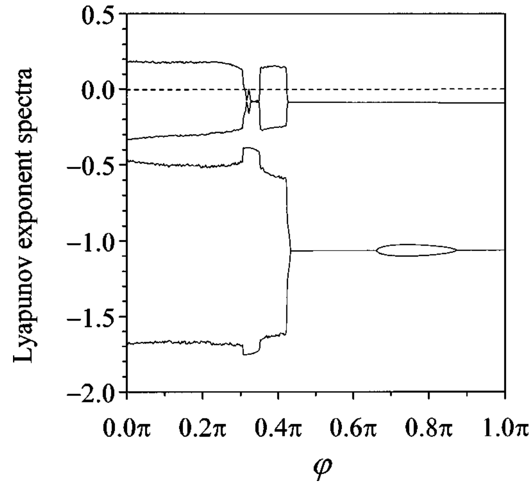

FIG. 6. Lyapunov exponent spectra vs φ for $C=1$.

increase of the coupling strength C, for the weak coupling case, S decreases exponentially [Fig. 8(b)]; but for strong coupling, $S(0)$ is proportional to C^{-1} [Fig. 8(a)]. This indicates that strong coupling strength can suppress the deviation induced by the phase difference. Therefore, in a rough manner of speaking, we can take $x(t)\approx u(t)$, $y(t)\approx v(t)$ for the very strong coupling. Define $s_x(t)=[x(t)+u(t)]/2$, and $s_y(t)=[y(t)+v(t)]/2$. Equation (3a) adds Eq. (3c), and Eq. (3b) adds Eq. (3d), so

$$\begin{aligned}\dot{s}_y&=s_y,\\ \dot{s}_y&=-\alpha s_y+s_x-s_x^2+\beta\cos(\varphi/2)\cos(\omega t+\varphi/2).\end{aligned}\tag{6}$$

Equation (6) is similar to Eq. (2), we find $\beta\sim\beta\cos(\varphi/2)$, and time is previous to $\varphi/2$ in the strongly coupled case. The result is qualitatively verified in Fig. 9,

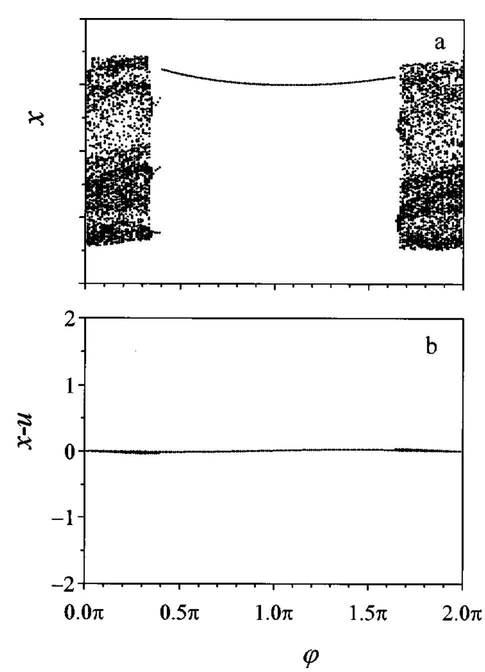

FIG. 7. Bifurcation diagram for a stronger coupling strength $C=5$ showing the Poincare map vs the initial phase difference φ. (a) x vs φ. (b) $x-u$ vs φ.

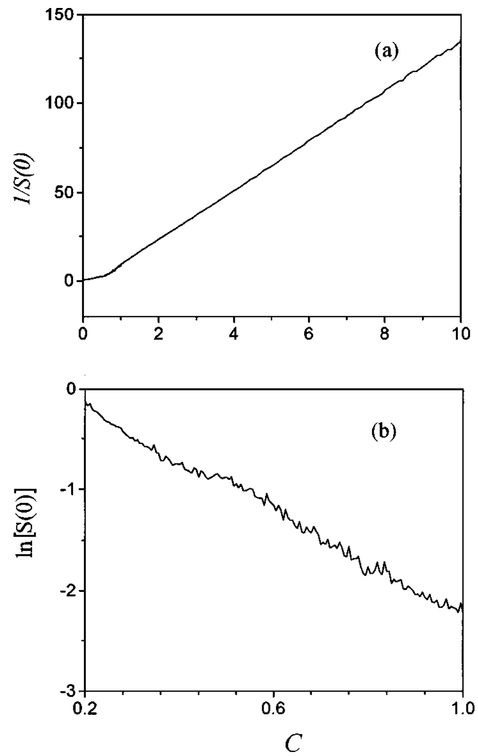

FIG. 8. The similarity function $S(0)$ vs the coupling strength C for the phase difference $\varphi=0.2\pi$. (a) For strong coupling strength, $S(0)$ is proportional to C^{-1}. (b) For weak coupling case, $S(0)$ exponentially decreases.

where the bifurcation diagram is plotted versus β for Eq. (2). Compare Fig. 9 with Fig. 7; they have a very similar bifurcation diagram, so the strong coupling strength makes two coupled systems identical. On the contrary, the phase difference prevents this tendency and enlarges the chaotic region (compare Fig. 3 with Fig. 7). For the large enough phase shift, it even eliminates chaos, and turns chaotic motion into periodic, which corresponds to small β. This conclusion is obvious if we consider Eq. (6).

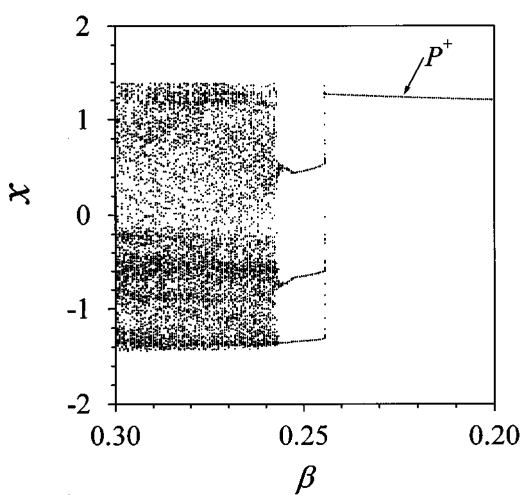

FIG. 9. The bifurcation diagram of Eq. (2) showing the Poincare map vs the amplitude β. P^+ indicates the period-1 orbit lying in the region $x>0$.

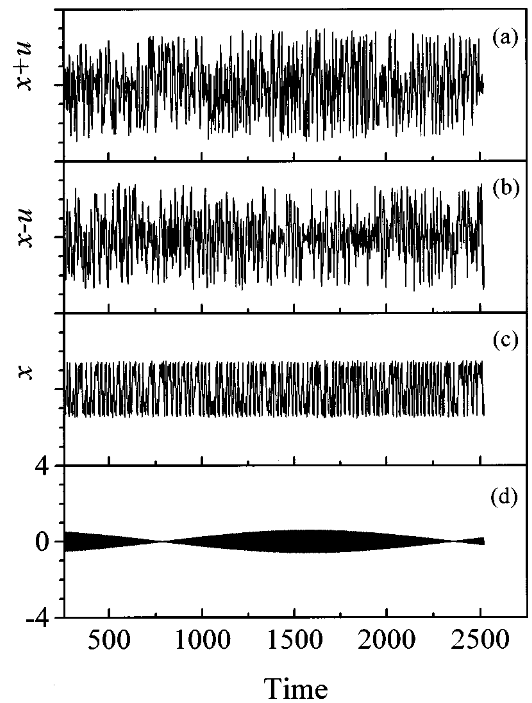

FIG. 10. The temporal dynamics of the variables for $C=0.05$ and $\Delta\omega=0.004$, (a) $x+u$, (b) $x-u$, (c) x and (d) the sum of the two amplitudes. It indicates two chaotic oscillators are not synchronous.

III. DETUNING OF FREQUENCIES

In many practical situations, the frequencies of two externally driven forces usually have small detuning. For small enough coupling strength, two chaotic oscillators are almost

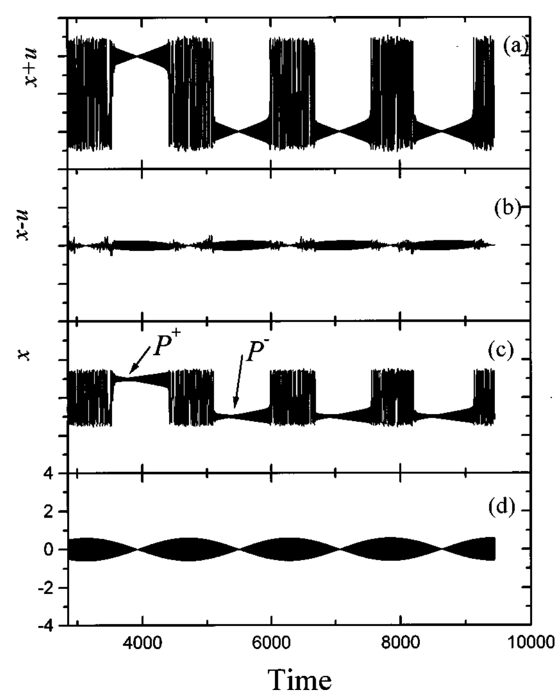

FIG. 11. The temporal dynamics of the variables for $C=1$ and $\Delta\omega=0.004$, (a) $x+u$, (b) $x-u$, (c) x, and (d) the sum of amplitude of two forces. This indicates that two chaotic oscillators synchronize when $\cos[\varphi(t)]=\pm1$. The sequence of regular motion P^+ and P^- is stochastic.

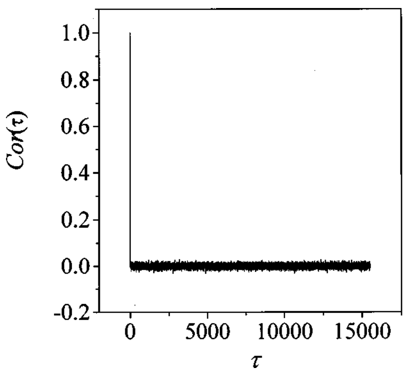

FIG. 12. The correlation function Cor(τ) of the sequence of P^+ and P^- for calculating 15 500 T.

uncorrelated. Figure 10 shows the temporal dynamics of the variable $x+u$ [Fig. 10(a)], $x-u$ [Fig. 10(b)], x [Fig. 10(c)] and sum of two externally driving forces [Fig. 10(d)] for $C=0.05$. In this case, the initial phase difference is not important. In the follows, we set $\varphi=0$, and $\Delta\omega=0.004$. For large coupling strength, there are several interesting phenomena, stochastic breathing, bifurcation delay [21,22], and stochastic transition.

Figure 11 shows the temporal dynamics of the variable $x+u$ [Fig. 11(a)], $x-u$ [Fig. 11(b)], x [Fig. 11(c)] and sum of two externally driving forces [Fig. 11(d)]. It shows that the regular and chaotic motions periodically appear with a new period $T=2\pi/\Delta\omega$, but the sequence of regular motion (P^+ or P^-) is stochastic. We refer this intermittency as to *stochastic breathing*. A similar result has been reported in Ref. [16], the authors call the periodic appearance of regular and chaotic motion as ''breathing.'' Reference [15] explains this result by geometrical resonance theory. The difference between Ref. [16] and our result is the two regular motions (P^+ or P^-) are stochastic appearance. The physical interpretation of our results is made in terms of a time dependent phase difference $\varphi(t)=\Delta\omega t$, which comes from the small difference of frequencies. The term $\beta\cos(\varphi/2)\cos(\omega t+\varphi/2)$ in Eq. (4) is changed to $\beta\cos(\varphi(t)/2)\cos[\omega t+\varphi(t)/2]$. It shows the sum of amplitudes of driving force slowly varies with time, so the motion is very like the static bifurcation diagram in Fig. 7. Suppose at some time $\cos[\varphi(t)/2]=1$, the motion is chaotic. As time increases, $\varphi(t)$ very slowly changes over the time interval, and the motion evolves chaotically up to a time t_1 for which $\varphi(t_1)$ is above the bifurcation point φ in Fig. 7. Chaotic motion has same chance to visit the two basins, when it comes stochastically into the basin of P^+ (or P^-), the regular motion will be P^+ (or P^-). To verify this results, we calculate 15 500 T showing 49.77% for P^+ and 50.23% for P^-, its correlation function Cor(τ) is plotted in Fig. 12. Therefore, appearance of P^+ (or P^-) is stochastic.

Now we discuss bifurcation delay [21] and from chaotic motion to regular motion. The bifurcation delay has been observed in various physical systems [22], which an external parameter slowly varies in time, and was analyzed in Ref. [23]. The stochastic transition is reported in our paper. In our model, there is not a slowly time dependent external parameter, however, due to the frequency detuning, there exists the aforementioned time dependent phase difference $\varphi(t)$, so we expected that bifurcation delay can be observed. Figure 13 is a Poincare section for $\varphi'[\varphi'=\varphi(t)\text{mod}2\pi]$. Comparing Fig. 7 with Fig. 13, we recognize that the bifurcation delay occurs from chaos(period) to period(chaos). The symmetry in Fig. 3 is broken in Fig. 13. Because φ' varies slowly, and just passes through the bifurcation point φ_c corresponding to the static bifurcation diagram, the oscillators will remain for some time before arriving at the new motion state. So bifurcation occurs for $\varphi'>\varphi_c$. However, the transition from chaotic motion to regular motion stochastically distributes in the small region between 0.50π and 0.63π. It can be qualitatively explained as follows. Before transition occurs, the motion is chaotic. When φ' varies and passes through a bifurcation point corresponding to the static bifurcation diagram, the system state can be regarded as initial conditions for period orbit. The distribution of the initial conditions is stochastic. If an initial state is close to the periodic orbit, the motion quickly tends to it; otherwise, the motion can be regarded as transient chaos, which needs a longer time to converge to the periodic orbit. It induces the stochastic transition.

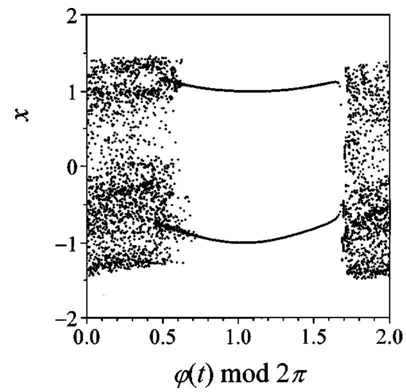

FIG. 13. The Poincare map for $\varphi(t)\text{mod}(2\pi)$ for $\Delta\omega=0.004$ and $C=5$. When compared with Fig. 3, it is obvious that bifurcation delay occurs, the motion from chaos to period is stochastic, and the bifurcation point is stochastically distributed between 0.5π and 0.62π.

IV. CONCLUSION

In this paper, we have considered the phase effect of the two mutually coupled Duffing oscillators. The phase difference plays an important role. It has two effects: first, it is almost equivalent to the change of the amplitude of the driving force for the single oscillator. Second, it destroys the complete synchronization of two oscillators. Therefore, further increase of the phase difference even eliminates chaos and leads the coupled oscillators to periodic motion. The coupling term suppresses the variation induced by the phase difference, brings about lag synchronization, and does not change the dynamics. For the case of the large coupling strength, we have found stochastic breathing, bifurcation delay, and stochastic transition for small frequency detuning. We report on stochastic breathing and transition. For very strong coupling and small phase difference, the variation of the two states practically tends to zero, and the motion of the coupled systems can be roughly described by the single oscillator. For the small detuning of the two driving frequencies, the quasistatic drift in the phase appears; it makes the motion to evolve under the corresponding local almost adiabatic invariant.

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